JOURNAL OF COMPUTATIONAL PHYSICS 93, 325–335 (1991)

On the Nature of Boundary Conditions for Flows with Moving Free Surfaces

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Received July 24, 1989; revised January 10, 1990

We consider small perturbations of plane parallel flow between a wall and a moving free surface. The problem is posed on a rectangle with inflow and outflow boundaries. The usual boundary conditions are posed at the wall and the free surface, and the fluid satisfies the Navier–Stokes equations. We examine the nature of boundary conditions which can be imposed at the inflow and outflow boundaries in order to yield a well-posed problem. This question turns out to be more delicate than is generally appreciated. Depending on the precise situation and on the regularity required of the solution, boundary conditions at just one or both endpoints of the free surface need to be imposed. For example, we show that if the velocities at the inflow and outflow boundaries are prescribed, then the position of the free surface at the inflow boundary can be prescribed, but not at the outflow if an H^1-solution is desired. Numerical simulations with the FIDAP package are used to illustrate our analytical results.

1. INTRODUCTION

For flows with free surfaces terminating at walls, the prescription of the contact point or the contact angle has long been accepted as a correct boundary condition. In recent years, a number of rigorous existence and uniqueness theorems have been established for problems of this kind; we refer to [2, 3] for a review and for references to previous literature: these concern steady problems with free surfaces terminating at walls.

Many numerical simulations of practically relevant flows, however, involve free surfaces which terminate not at a wall, but at a computational inflow or outflow boundary. An example is the free-surface problem that arises in coating flows, for which a portion of the domain is truncated and solved. Another example is layered flow in a channel of finite length, with interfaces: this raises the question of what conditions to pose at inflow and outflow boundaries. What typifies these examples is a numerical truncation of the flow domain. This type of situation does not seem to have been analyzed nearly as well. Since in- and outflow boundaries are mathematical artifacts, the nature of boundary conditions to be imposed cannot be guessed from physical reasoning.

In the following, we consider a model problem which is easy to analyze and at

0021-9991/91 $3.00

the same time provides a prototype for the more complex problems which might arise in real applications. We study small perturbations of a plane parallel flow bounded by a wall at the bottom and a moving free surface at the top. We investigate the linearized problem. We shall demonstrate that in some situations well-posed problems are obtained if the position of the free surface is prescribed only at the upstream boundary, not at both boundaries. We believe that this may explain some pathological behavior which was observed in attempts to simulate flows of this type using the FIDAP package [1]. These numerical results are discussed in Section 3 below.

The equations of interest are the Stokes equations (the inertial nonlinearities in the Navier–Stokes equations contribute only terms of lower differential order which are not important in the analysis that follows)

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \eta\, \Delta \mathbf{v} - \nabla p, \qquad \operatorname{div} \mathbf{v} = 0, \tag{1}$$

in the two-dimensional domain given by $0 < x < L$, $0 < y < M + h(x)$. The bottom is a moving wall with the no-slip condition,

$$\mathbf{v}(x, 0) = (U, 0), \tag{2}$$

and the top is a free surface. With $\mathbf{n} = (-h'(x), 1)/\sqrt{1+h'^2}$ and $\mathbf{t} = (1, h'(x))/\sqrt{1+h'^2}$ denoting the unit normal and tangent, we have the boundary conditions of continuity of shear stress and the balance of normal stress by surface tension,

$$\eta\mathbf{n}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}})\mathbf{t} = 0, \qquad \eta\mathbf{n}(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}})\mathbf{n} - p = S\frac{h''}{(1+h'^2)^{3/2}}, \tag{3}$$

and the kinematic free-surface condition

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = v. \tag{4}$$

Equations (1)–(4) have the exact solution $h = 0$, $\mathbf{v} = (U, 0)$, $p = 0$. We consider small perturbations of this exact solution, which are caused by adding forcing terms to (1), (3), and (4). We linearize the equations and take Laplace transforms with respect to time. The perturbations then satisfy the Stokes equations

$$\rho\lambda u = \eta\Delta u - \frac{\partial p}{\partial x} + f_1,$$

$$\rho\lambda v = \eta\Delta v - \frac{\partial p}{\partial y} + f_2,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

on the unperturbed domain $0<x<L$, $0<y<M$. Here, f_1 and f_2 are the given forcing terms. At the bottom wall $y=0$, we have the no-slip condition

$$u=v=0, \tag{6}$$

and at $y=1$, the linearization of (3) and (4) yields

$$\begin{aligned} \eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)&=f_3,\\ 2\eta\frac{\partial v}{\partial y}-p&=Sh''+f_4,\\ \lambda h+Uh'&=v+f_5, \end{aligned} \tag{7}$$

where f_3, f_4, and f_5 are the given forcing terms. At the inflow and outflow boundaries $x=0, L$ we shall impose either the tangential velocity

$$v=0 \tag{8a}$$

or the shear stress

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0, \tag{8b}$$

and either the normal velocity

$$u=0 \tag{9a}$$

or the normal stress

$$2\eta\frac{\partial u}{\partial x}-p=0. \tag{9b}$$

In the following, we discuss the well-posedness of the problem (5)–(9) in appropriate function spaces. In order to obtain a well-posed problem, we need to impose boundary conditions on $h(x, t)$. The nature of these boundary conditions is the principal issue of the paper. We first give some heuristic considerations. If $\lambda=0$ and $U=0$, we can first ignore the second equation of (7) and use the remaining two equations as boundary conditions for the Stokes problem. After solving the Stokes problem, the second equation in (7) is used to obtain h. Clearly, this requires two boundary conditions on h, e.g., h may be prescribed at each endpoint. If λ and U are not zero, we may hope to treat the terms λh and Uh' as a perturbation. If, on the other hand, $S=0$, then we can use the first two equations of (7) as boundary conditions for the Stokes problem and then recover h from the third equation. Since the third equation of (7) is first order in h, only one boundary condition is required. At first it may appear that the term Sh'' is necessarily a singular perturbation, because it involves the highest derivative of h. However, this appearance is deceiving. The term Sh'' in (7) appears in the same equation as derivatives of the velocity, while the term Uh' appears together with the velocity. Hence the two terms are formally of the same order. Whether either of these terms (or both) may be treated as a regular perturbation requires a careful analysis and will depend on the function spaces chosen for the analysis.

2. Well-posedness for the Linearized Equations

We shall consider (5)–(9) in the context of weak (variational) solutions. Let Ω denote the domain $(0, L) \times (0, M)$ and let Γ denote the top boundary. For the moment, we shall consider only the boundary condition (8a). We seek velocities in the space X_b or X_a, defined as

$$X_b = \{\mathbf{v} \in (H^1(\Omega))^2 \mid \operatorname{div} \mathbf{v} = 0,\ \mathbf{v}(\cdot, 0) = \mathbf{0},\ v(0, \cdot) = v(L, \cdot) = 0\}, \tag{10b}$$

if boundary condition (9b) applies. Here, the vertical components of the velocity and normal stress vanish. The subset of X_b where boundary condition (9a) applies is

$$X_a = \{\mathbf{v} \in X_b \mid u(0, \cdot) = u(L, \cdot) = 0\}. \tag{10a}$$

In the following X denotes X_a or X_b, whichever is appropriate.

The equations have the weak form

$$\int_\Omega \rho\lambda \mathbf{v} \cdot \bar{\mathbf{\Phi}} + \eta(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}) : \nabla\bar{\Phi}\, dx\, dy = (\mathbf{F}, \mathbf{\Phi}) + S \int_0^L h''\bar{\psi}(x, M)\, dx, \tag{11}$$

where the notation $A:B$, with matrices A and B, denotes $\sum_{i,j} A_{ij} B_{ij}$. The vector $\mathbf{\Phi} = (\phi, \psi)$ is a test function and is an arbitrary function in X, where ϕ and ψ are functions of x and y. $(\mathbf{F}, \mathbf{\Phi})$ denotes the expression

$$\int_\Omega f_1\bar{\phi} + f_2\bar{\psi}\, dx\, dy + \int_\Gamma f_3\bar{\phi} + f_4\bar{\psi}\, dx, \tag{12}$$

where the f_i appear in (5) and (7). In the usual manner, we take (11) with $\mathbf{F} \in X'$ (the dual space of X) as the generalized interpretation of our equations. In addition to (11), we still have the kinematic free-surface condition

$$\lambda h + Uh' = v + f_5. \tag{13}$$

We now distinguish the following three cases:

(i) $S = U = 0$,

(ii) $U = 0,\ S > 0$,

(iii) $U > 0,\ S \geqslant 0$.

Our main result is the following.

THEOREM. (i) *Assume* $S=U=0$. *Then for every* λ *with* $\operatorname{Re}\lambda\geqslant 0$ *and* $\lambda\neq 0$, *and every* $\mathbf{F}\in X'$, $f_5\in H^{1/2}(0,L)$, *Eqs.* (11) *and* (13) *have a unique solution such that* $\mathbf{v}\in X$, $h\in H^{1/2}(0,L)$.

(ii) *Assume* $U=0$, $S>0$. *Then for every* λ *with* $\operatorname{Re}\lambda\geqslant 0$ (*if boundary condition* (9a) *applies, we exclude the case* $\lambda=0$), *and every* $\mathbf{F}\in X'$, $f_5\in H_{00}^{1/2}(0,L)$, *Eqs.* (11) *and* (13) *have a unique solution such that* $\mathbf{v}\in X$, $h\in H^{3/2}(0,L)\cap H_0^1(0,L)$.

(iii) *Assume* $U>0$, $S\geqslant 0$. *Then for every* λ *with* $\operatorname{Re}\lambda\geqslant 0$ *and every* $\mathbf{F}\in X'$, $f_5\in H^{1/2}(0,L)$, *Eqs.* (11) *and* (13) *have a unique solution such that* $\mathbf{v}\in X$, $h\in H^{3/2}(0,L)$, *and* $h(0)=0$.

We refer to [4] for the definition of the function spaces used here. In cases (ii) and (iii), the integral on the right-hand side of (11) is interpreted by duality between the space $H_{00}^{1/2}(0,L)$ and its dual, which we denote by $H_*^{-1/2}(0,L)$. We note that the results for the three cases are quite different. In particular, the term Uh' in (13) cannot be considered a regular perturbation even if U is small. We note also that part (iii) makes no distinction between the cases $S=0$ and $S>0$. As we shall see in the proof, however, there is a difference in the estimates which apply for $|\lambda|\to\infty$.

We now give the proof of the theorem. Case (i) is straightforward. If this case applies, then h disappears from Eq. (11), which is now simply the Stokes problem. Equation (11) is uniquely solvable for $\operatorname{Re}\lambda\geqslant 0$, and the solution satisfies the estimate

$$\|\mathbf{v}\|_X+\sqrt{|\lambda|}\,\|\mathbf{v}\|_{L^2}\leqslant C\,\|\mathbf{F}\|_{X'},\tag{14}$$

which is obtained by setting $\mathbf{\Phi}=\mathbf{v}$ in (11). If $\lambda\neq 0$, we can then solve (13) for h and we have

$$|\lambda|\,\|h\|_{H^{1/2}}\leqslant\|f_5\|_{H^{1/2}}+C\,\|\mathbf{v}\|_X.\tag{15}$$

For case (ii), we first consider $\lambda=0$. In this case, (13) yields $v=-f_5$ (if (9a) applies, then this is consistent with the divergence condition only if f_5 has zero average). We combine this equation with (11), where $\mathbf{\Phi}$ is restricted to satisfy $\psi|_\Gamma=0$. The problem thus obtained is a Stokes problem which is uniquely solvable for $\mathbf{v}$. After inserting $\mathbf{v}$ into (11), we can determine $h''\in H_*^{-1/2}(0,L)$ (if (9a) applies, then all test functions $\mathbf{\Phi}$ satisfy $\int_0^L\psi(x,M)\,dx=0$ and hence (11) determines h'' only up to a constant). From h'' and the imposed boundary conditions $h(0)=h(L)=0$, we can determine $h\in H^{3/2}(0,L)$. With Eqs. (11) and (13), we can associate a linear operator L from $X\times H^{3/2}(0,L)\cap H_0^1(0,L)$ into $X'\times H_{00}^{1/2}(0,L)$, which maps $(\mathbf{v},h)$ to $(\mathbf{F},f_5)$. We have just shown that, for $\lambda=0$ and boundary condition (9b), L is invertible, while for (9a) the nullity and deficiency are both equal to 1. The operator for $\lambda\neq 0$ is a compact perturbation and is hence Fredholm of index zero. Hence it suffices to show uniqueness of solutions; existence follows automatically. We now set $\mathbf{\Phi}=\mathbf{v}$ in (11) and we use (13) in the term in the integral on the right-hand side of (11). This yields, after an integration by parts,

$$\rho\lambda\,\|\mathbf{v}\|_{L^2}^2+\eta\int_\Omega (\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}):\nabla\bar{\mathbf{v}}\,dx\,dy+S\bar{\lambda}\int_0^L |h'|^2\,dx$$
$$=(\mathbf{F},\mathbf{v})-S\int_0^L h''\bar{f}_5\,dx. \tag{16}$$

The uniqueness of solutions follows immediately.

We now turn to case (iii), and again we begin with $\lambda=0$. In this case, (13) yields $h''=(v'+f_5')/U$, which we insert into (11). This yields

$$\eta\int_\Omega (\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}):\nabla\bar{\mathbf{\Phi}}\,dx\,dy$$
$$=(\mathbf{F},\mathbf{\Phi})+\frac{S}{U}\int_\Gamma \frac{\partial v}{\partial x}\bar{\psi}\,dx+\frac{S}{U}\int_0^L f_5'(x)\,\bar{\psi}(x,M)\,dx. \tag{17}$$

For $S=0$, we have the Stokes problem, which is uniquely solvable. If $S/\eta U$ is sufficiently small, we also get unique solvability by regular perturbation theory (the mapping $v\to\partial v/\partial x\,|_\Gamma$ is continuous from X to $H_*^{-1/2}(0,L)$). Moreover, if we set $\mathbf{\Phi}=\mathbf{v}$ and take the real part in (17), we obtain

$$\eta\int_\Omega (\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}):\nabla\bar{\mathbf{v}}\,dx\,dy=\mathrm{Re}(\mathbf{F},\mathbf{v})+\frac{S}{U}\mathrm{Re}\int_0^L f_5'(x)\,\bar{v}(x,M)\,dx. \tag{18}$$

From this, we obtain the a priori estimate

$$\|\mathbf{v}\|_X\leqslant C(\|\mathbf{F}\|_{X'}+\|f_5\|_{H^{1/2}}). \tag{19}$$

This implies that the operator corresponding to (17) is injective and has closed range. The continuity of the Fredholm index implies now that (17) is uniquely solvable for every S. Having found $\mathbf{v}$, we can determine h from (13) if we impose one boundary condition.

The case $\lambda\neq 0$ is again a compact perturbation of $\lambda=0$. Hence it suffices to show uniqueness of solutions. To this purpose, we set $\mathbf{\Phi}=\mathbf{v}$ and use (13) twice to transform the integral on the right-hand side of (11) as

$$\int_0^L h''\bar{v}\,dx=\frac{1}{U}\int_0^L (-\lambda h'+v'+f_5')\bar{v}\,dx$$
$$=\frac{\lambda}{U}\int_0^L h'\bar{f}_5\,dx-\frac{|\lambda|^2}{U}\int_0^L h'\bar{h}\,dx-\lambda\int_0^L |h'|^2\,dx$$
$$+\frac{1}{U}\int_0^L (v'+f_5')\bar{v}\,dx. \tag{20}$$

By inserting into (11) and taking real parts, we obtain

$$\rho(\mathrm{Re}\,\lambda)\,\|\mathbf{v}\|_{L^2}^2+\eta\int_\Omega(\nabla\mathbf{v}+(\nabla\mathbf{v})^T):\nabla\bar{\mathbf{v}}\,dx\,dy+S(\mathrm{Re}\,\lambda)\int_0^L|h'|^2\,dx+\frac{S\,|\lambda|^2}{U}\,|h(L)|^2$$

$$=\mathrm{Re}(\mathbf{F},\mathbf{v})+\frac{S}{U}\mathrm{Re}\int_0^L\lambda h'\bar{f}_5+f'_5\bar{v}\,dx. \tag{21}$$

Uniqueness of solutions follows immediately.

We shall next consider the boundary conditions (8b) and (9a). We consider only the case $\lambda=0$. The space X should be defined as

$$X=\{\mathbf{v}\in(H^1(\Omega))^2\mid \mathrm{div}\,\mathbf{v}=0,\ \mathbf{v}(\cdot,0)=\mathbf{0},\ u(0,\cdot)=u(L,\cdot)=0\}. \tag{22}$$

We shall consider data $F\in X'$, $f_5\in H^{1/2}(0,L)$ and seek solutions $\mathbf{v}\in X$, $h\in\{g\in H^{3/2}(0,L)\mid g''\in H^{-1/2}(0,L),\ g(0)=g(L)=0\}$. If $U=0$, $S>0$, we proceed analogously as in case (ii) above. We obtain $v=-f_5$ from (13), and this equation together with (11) (where $\mathbf{\Phi}$ is restricted by the condition $\psi(\cdot,M)=0$) forms a Stokes problem which is uniquely solvable. Compatibility with the divergence conditions requires that the average of f_5 must be zero. From (11) we can then determine h'' up to a constant, and from h'' and the boundary conditions $h(0)=h(L)=0$ we can determine h. Hence the Fredholm index of the operator in the given function spaces is zero. Since the mapping $h\to h'$ is continuous from $H^{3/2}$ to $H^{1/2}$, we can treat the term Uh' as a regular perturbation. Hence, at least for large values of the dimensionless surface tension parameter $S/\eta U$, we have Fredholm index zero if two boundary conditions are imposed on the free surface. We note that it is not possible to start from $S=0$, $U>0$ and treat the term Sh'' as a regular perturbation. This is because the mapping $h\to h''$ is continuous from $H^{3/2}$ only into $H_*^{-1/2}$, but not into $H^{-1/2}$.

This discussion seems to indicate that, when the shear stress rather than the vertical velocity is prescribed at the inflow and outflow boundaries, boundary conditions are required at both endpoints of the free surface; namely, one boundary condition is required for the free surface height h at each end. However, we shall see that the issue is more subtle. Let us consider the special case of (5)–(7), (8b), (9a), where $\lambda=0$ and, in addition, $f_1=f_2=0$. We now look for solutions of higher regularity. Let s be any number such that $1<s<2$. The following lemma holds.

LEMMA. (a) *Let* $U=0$, $S\neq 0$. *Then for every* $f_3,f_4\in H^{s-3/2}(0,L)$, $f_5\in H^{s-1/2}(0,L)$ *with* $\int_0^L f_5(x)\,dx=0$, *there is a unique* $\mathbf{v}\in H^s(\Omega)$ *and a one-parameter family of* $h\in H^{s+1/2}(0,L)\cap H_0^1(0,L)$.

(b) *Let* $U\neq 0$, $S=0$. *Then for every* $f_3,f_4\in H^{s-3/2}(0,L)$, $f_5\in H^{s-1/2}(0,L)$, *there is a unique solution* $\mathbf{v}\in H^s(\Omega)$, $h\in\{g\in H^{s+1/2}(\Omega)\mid h(0)=0\}$.

The proof is straightforward. In case (a), we proceed as above, in case (b), we proceed in analogy to case (iii) of the theorem. The regularity result required for the Stokes problem is easily obtained, because the boundary conditions which we have imposed are consistent with extending u as an odd periodic function and v and p as even periodic functions of x with period $2L$. Hence we only need to consider the Stokes problem with periodic data, without having to concern ourselves with investigating corner singularities.

We can now start from part (a) of the lemma and treat the term Uh' as a regular perturbation. However, the second derivative operator is continuous from $H^{s+1/2}$ into $H^{s-3/2}$, and hence we can also start from part (b) and treat the term Sh'' as a regular perturbation! Hence, if the capillarity number $\eta U/S$ is small, we obtain an operator of index zero by imposing two boundary conditions on h, but if $\eta U/S$ is large, we obtain an operator of index zero if we impose only one boundary condition. In between, there must be a critical value of $\eta U/S$ where the operator is not Fredholm with either number of boundary conditions. This critical value depends on s. Hence the number of boundary conditions required for the endpoints of the free surface depends on the parameters of the problem and also on the function space in which we require the solution to be. Clearly, a complete analysis would require an investigation of the behavior of solutions at the corners. The asymptotic behavior at the corners is discussed in [5].

Remark. There are also situations of practical interest where one end of the free surface is attached to a wall, while the other end is at an inflow or outflow boundary. This situation remains to be investigated.

3. Numerical Results

The numerical simulations were done on the full Navier–Stokes system using the fluid dynamics package (FIDAP) [1]. Nine-node quadratic quadrilateral elements are used for the interior, and three-node quadratic free-surface elements are used. The mixed velocity–pressure formulation with discontinuous pressure approximation is employed. The solution at each timestep of the nonlinear time-dependent problem is solved with the Newton–Raphson iterative method, which is accelerated (relaxed) by means of an acceleration factor of 0.5, together with the backward Euler time integration scheme. The initial timestep is 0.01, and successive timesteps are chosen adaptively.

Simulations are carried out on the domain $0 < x < 4\pi/3$, $0 < y < 1 + h(x, t)$, where initially we have $h(x, 0) = 0.1 \sin(3x)$. The mesh size is 27 by 13 nodes. The viscosity, density, and surface tension coefficient are all set equal to one. The bottom wall $y = 0$ moves with unit velocity, and the top boundary is a free surface. We note that the only dimensionless parameter which is relevant in the analysis above is the capillarity number $\eta U/S$, which is unity for the parameters used in the computation.

A number of simulations are performed with velocities $u = 1$, $v = 0$ prescribed at both the left and right boundaries. Initial conditions are generated in two different ways. For the first, we simply set the velocity equal to $(1, 0)$ throughout the whole domain. For the second, we solve the Stokes problem with prescribed velocity $(1, 0)$ at the bottom, left, and right boundaries, together with the free surface conditions (but with the position of the surface kept fixed) at the top boundary. For this second case, the initial data are compatible with the boundary conditions at the free surface, while for the first case they are not. There is little difference in the results between the two cases, and we only show those for the second case. Figure 1 shows the solutions obtained at various time steps when $h = 0$ is imposed at both endpoints of the free surface. Figure 2 shows results obtained when $h = 0$ is imposed only at the left endpoint and no condition on h is imposed at the right endpoint. In both simulations, the free surface is flattened by surface tension and the flow becomes uniform as would be expected. However, in Fig. 1 we can see unphysical wiggles in the free surface appearing near the outflow boundary. The wiggles are localized in one corner of the domain. Outside the small region where the wiggles appear, the plots in Figs. 1 and 2 are indistinguishable within graphical resolution. This matches up well with our analysis above which indicates that only the free surface position at the inflow boundary should be prescribed. The results displayed in Fig. 1 have been verified with a finer mesh. The only change was that the wiggles appeared on the scale of the new mesh.

In Fig. 3, we impose stress-free boundaries rather than velocity conditions at the inflow and outflow. Contact angles are prescribed at both endpoints of the free surface. These contact angles are chosen to be compatible with the initial free surface position, which is as above. The initial velocity field is generated by solving the Stokes problem with constant velocity $(1, 0)$ at the bottom, zero stresses at the left

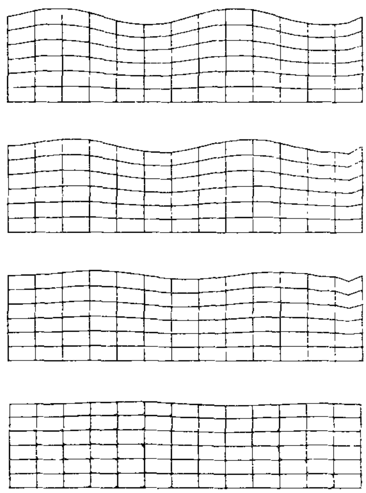

FIG. 1. The mesh is shown at timesteps 5, 10, 15, and 20, or times 0.1201, 0.4590, 0.9755, and 1.857. Note the presence of wiggles on the scale of the mesh where the free surface meets the outflow boundary.

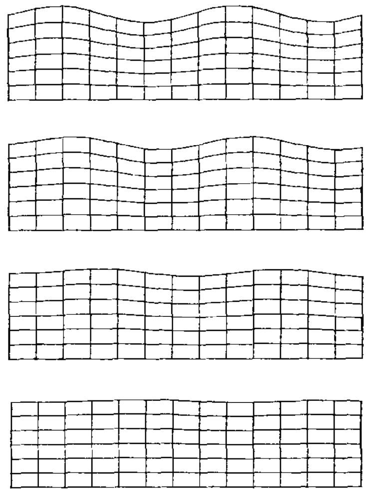

FIG. 2. The mesh is shown at timesteps 5, 10, 15, and 20, or times 0.1211, 0.4826, 1.069, and 2.341. No wiggles are present.

and right boundaries, and free surface conditions at the top boundary. Because of the prescribed non-zero contact angles, the solution does not approach uniform flow. On close inspection of the plots, one can see a very slight wiggle where the free surface meets the outflow boundary, but this is much less pronounced than in Fig. 1. This is consistent with the analysis above, which suggests only a mild singularity in the present case. It would be instructive to repeat the calculation with only one contact angle prescribed, but the FIDAP algorithm does not seem to make this possible.

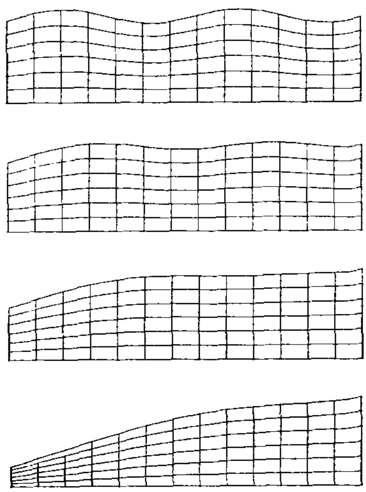

FIG. 3. The mesh is shown at timesteps 5, 10, 15, and 20, or times 0.0555, 0.2051, 0.5958, and 1.484. A very slight wiggle is present where the free surface meets the outflow boundary.

4. Conclusions

We investigated the motion of a fluid bounded by a moving wall at the bottom, a free surface on top and inflow and outflow boundaries at the left and right. The linearized problem for small perturbations of uniform flow was studied. If the vertical velocity and either the horizontal velocity or the normal stress is prescribed at the inflow and outflow boundaries, then a well-posed problem is obtained by prescribing only one boundary condition for the free surface position. If one tries to prescribe the free surface position at both endpoints, then, in general, there will not be solutions such that the velocity is in H^1. In numerical simulations, wiggles appear near the corner between the free surface and the outflow boundary; these wiggles disappear if the free surface position is prescribed only at the inflow boundary.

If the shear stress rather than the vertical velocity is prescribed at the inflow and outflow boundaries, the situation is different. Our analysis for this case is less complete, but it indicates that, in order to find a unique solution with velocity in H^s, $1 < s < 2$, it is sufficient to prescribe one boundary condition if the capillarity number is large, but two boundary conditions are required if the capillarity number is small.

Acknowledgments

This research was initiated while the authors were visiting the Institute for Mathematics and Its Applications at the University of Minnesota. Financial support from the IMA is gratefully acknowledged. Parts of the computations were carried out at the Minnesota Supercomputer Center. This research is supported by the Air Force Office of Scientific Research under Grant AFOSR-86-0085, DARPA under Grant F49620-87-C-0116, National Science Foundation Grants DMS-8796241, DMS-8720298, and DMS-8902166, and a grant at the Pittsburgh Supercomputer Center.

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